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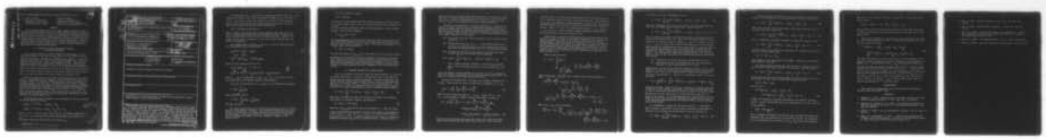

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ON THE REPRESENTATION OF NONLINEAR SYSTEMS WITH GAUSSIAN INPUTS

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ABSTRACT

An arbitrary nonlinear system with input a Gaussian process, which is such that its output process has finite second moments, admits two kinds of representations; the first in terms of a sequence of deterministic kernels and the second in terms of a single stochastic kernel. We consider here the identification of the sequence of deterministic kernels from the input and output processes, the representation of the system output when its input is a sample function of the Gaussian process, and the relationship of the sequence of kernels mentioned above to the Volterra expansion kernels when the system has a Volterra representation.

1. STOCHASTIC AND MULTIPLE WIENER INTEGRALS FOR GAUSSIAN PROCESSES

Let us first introduce our basic notation and terminology. We will consider throughout a zero mean Gaussian process $X = (X_t, t\in T)$ with covariance function $R(t,s)$, defined on a probability space $(\Omega,\mathcal{B},P)$. T will be an interval of the real line. $\mathcal{B}$ is usually taken to be $\mathcal{B}(X)$, the σ-field generated by the process X, or $\overline{\mathcal{B}}(X)$, the completion of $\mathcal{B}(X)$. There are two important Hilbert spaces associated to a Gaussian process. The nonlinear space of X, $L_2(X) = L_2(\Omega,\mathcal{B}(X),P)$, consists of all $\mathcal{B}(X)$-measurable random variables with finite second moment which are called (nonlinear) L_2-functionals of X. The linear space of X, $H(X)$, is the closed subspace of $L_2(X)$ spanned by X_t, $t\in T$, and its elements are called linear L_2-functionals of X.

The first useful notion in the study of the nonlinear space of a Wiener process is the Multiple Wiener Integral. This notion was first introduced by Wiener (1938), who termed it "Polynomial Chaos," and was redefined in a somewhat deeper way by Itô (1951). Itô showed that his multiple integrals of different degree have the important property of being mutually orthogonal and also presented their connection with the celebrated Fourier-Hermite expansion of L_2-functionals of Cameron and Martin (1947). In his important work on nonlinear problems Wiener (1958) reinterpreted the multiple Wiener integrals for a Wiener process in an extremely simple and intuitive way and made some interesting applications.

In [4] multiple Wiener integrals of the following two types are defined for general Gaussian processes:

$$I_p(f_p) = \int\cdots\int f(t_1,\ldots,t_p)dX_{t_1}\ldots dX_{t_p}$$

$$J_p(f_p) = \int\cdots\int f(t_1,\ldots,t_p)X_{t_1}\ldots X_{t_p}\, dt_1\ldots dt_p$$

where $p = 1,2,\ldots$, and we always write $\int$ for $\int_T$. We always assume that $X_{t_0} = 0$ a.s. for some $t_0\in T$ when dealing with integrals I_p, and that X is mean square continuous when dealing with integrals J_p.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An arbitrary nonlinear system with input a Gaussian process, which is such that its output process has finite second moments, admits two kinds of representations; the first in terms of a sequence of deterministic kernels and the second in terms of a single stochastic kernel. We consider here the identification of the sequence of deterministic kernels from the input and output processes, the representation of the system output when its input is a sample function of the Gaussian process, and the relationship of the sequence of kernels mentioned above to the Volterra expansion kernels when the system has a Volterra representation.

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The integrands $f(t_1,\ldots,t_p)$ belong to appropriate Hilbert spaces of "functions" defined on T^p, which are denoted by $\Lambda_2(\otimes^p R)$ for I_p, and $\lambda_2(\otimes^p R)$ for J_p. For instance $\lambda_2(R)$ is the completion of the set of all functions $f(t)$ on T which are such that the Riemann integral $R\iint f(t)f(s)\,R(t,s)dt\,ds$ exists and is finite with respect to the inner product

$$<f,g> = R\iint f(t)f(s)\,R(t,s)dt\,ds\ .$$

When $T = [a,b]$, $\lambda_2(R)$ contains all square integrable functions over T, as well as further interesting classes of functions (see [4]), but also "functions" with properties similar to those of delta functions. $\lambda_2(\otimes^p R)$ is defined similarly, and is isomorphic to the tensor product $\otimes^p \lambda_2(R)$ under the natural correspondence.

The multiple Wiener integrals J_p, $p=1,2,\ldots$, have the following properties ($f,g\in\lambda_2(\otimes^p R)$ and a,b real numbers):

$$J_p(af+bg) = aJ_p(f) + bJ_p(g)\ ,$$

$$J_p(f) = J_p(\tilde{f})\ ,$$

$$<J_p(f),\ J_p(g)>_{L_2(X)} = p!<\tilde{f},\tilde{g}>_{\lambda_2(\otimes^p R)}\ ,$$

where $\tilde{f}$ is the symmetric version of f,

$$<J_p(f),J_q(q)>_{L_2(x)} = 0 \quad \text{if } p \neq q\ ,$$

$$J_p(\phi_1^{\tilde{\otimes} p_1}\ \tilde{\otimes}\ldots\tilde{\otimes}\ \phi_k^{\tilde{\otimes} p_k}) = H_{p_1}(\int\phi_1(t)X_t dt)\ldots H_{p_k}(\int\phi_k(t)X_t dt)\ ,$$

where $\phi_1,\ldots,\phi_k$ are orthonormal in $\lambda_2(R)$, $p_1+\ldots+p_k = p$, H_p denote the Hermite polynomials, and $\tilde{\otimes}$ denotes symmetric tensor product.

Also every L_2-functional θ of X, $\theta\in L_2(X)$, has an orthogonal development

$$\theta = E(\theta) + \sum_{p=1}^{\infty} J_p(f_p)$$

where $f_p\in\lambda_2(\otimes^p R)$, and if

$$\theta - E(\theta) = \sum_{p=1}^{\infty} J_p(f_p) = \sum_{p=1}^{\infty} J_p(g_p)$$

then $\tilde{f}_p = \tilde{g}_p$, $p=1,2,\ldots$.

The second useful notion in the study of the nonlinear space of a Wiener process is the stochastic integral. The stochastic integral was first introduced by Itô (1944) for the Wiener process and later generalized by Meyer for martingales. Every L_2-functional of a Wiener process has a representation as a stochastic integral, where the integrand is adapted to the Wiener process.

In [4] a stochastic integral

$$I(f) = \int f(t)dX_t$$

is defined for $L_2(X)$-valued "functions" f(t) belonging to the Hilbert space $\Lambda_{2;L_2(X)}(R)$, which is defined in a way similar to $\Lambda_2(R)$ and is isomorphic to the tensor product $\Lambda_2(R)\otimes L_2(X)$ under the correspondence $\phi\xi \leftrightarrow \phi\otimes\xi$, $\phi\epsilon\Lambda_2(R)$, $\xi\epsilon L_2(X)$. The stochastic integral

$$I: \Lambda_{2;L_2(X)}(R) \to L_2(X)$$

is an unbounded densely defined closed linear map with range the set of all zero mean random variables in $L_2(X)$ (for further properties and some evaluations of the stochastic integral see [4]). Thus every L_2-functional θ of X, $\theta\epsilon L_2(X)$, admits the representation

$$\theta = E(\theta) + \int f(t)dX_t$$

for some (non-unique) $f\epsilon\Lambda_{2;L_2(X)}(R)$ in the domain of I, and f may be taken to be adapted to X ("adapted to" meaning "measurable with respect to the past of"). Notice that as shown in [4], the stochastic integral becomes norm preserving when restricted to nonanticipatory integrands, but it is not yet known which L_2-functionals admit nonanticipatory stochastic integral representations ("nonanticipatory" meaning "independent of the future increments of X").

2. NONLINEAR SYSTEMS WITH GAUSSIAN INPUTS

Let θ be a nonlinear system with input the Gaussian process $X = \{X_t, t\epsilon T\}$ and output the process $Y = \{Y_t, t\epsilon T\}$ (the parameter sets of the input and output processes could of course be distinct). The only assumption we make on the nonlinear system without further notice is that it is such that the process Y is of second order, i.e. each Y_t is an L_2-functional of X. Then from Section 1, we have two representations of the system θ for the input X. The first is a series representation in terms of multiple Wiener integrals

$$Y_t = E(Y_t) + \sum_{p=1}^{\infty} \int\cdots\int f_p(t;t_1,\ldots,t_p)X_{t_1}\ldots X_{t_p}dt_1\ldots dt_p \tag{1}$$

where $f_p(t;\cdot)\epsilon\Lambda_2(\otimes^p R)$ may (and will from now on) be taken to be symmetric. The second is a stochastic integral representation

$$Y_t = E(Y_t) + \int f(t;s)X_s ds \tag{2}$$

where $f(t;\cdot)\epsilon\Lambda_{2;L_2(X)}(R)$ may be taken to be adapted to X. Thus the system θ can be represented for the input X either by the sequence of deterministic kernels $f_p(t;t_1,\ldots,t_p)$, p=1,2,..., as in (1), or by the single stochastic kernel f(t,s) as in (2). It should be emphasized that both, the sequence of deterministic kernels $\{f_p\}$ and the stochastic kernel f, depend not only on the system θ but also on the input Gaussian process X. Thus distinct input processes will always produce distinct stochastic kernels (unless of course the system is linear) in representation (2), and will in general produce distinct sequences of deterministic kernels in representation (1).

From now on we concentrate on the representation (1). Notice that expansion (1) looks very much like a Volterra kernel expansion, with the important difference that the multiple integrals are multiple Wiener rather than Lebesgue integrals; it could thus be considered as a stochastic Volterra kernel expansion. Several system synthesis or identification problems naturally suggest themselves at this point:

(P1) Knowing the input and output processes X and Y, find the kernels f_p, $p=1,2,\ldots$.

(P2) Knowing the kernels f_p, $p=1,2,\ldots$, in the representation of the system for the Gaussian input X, can one represent the output of the system to another Gaussian input or to a deterministic input?

(P3) Assuming that the system θ when acting on deterministic inputs in, say, $L_2[a,b]$ has a Volterra kernel expansion

$$y(t) = k_0(t) + \sum_{p=1}^{\infty} L\int\cdots\int k_p(t;t_1,\ldots,t_p)x(t_1)\ldots x(t_p)dt_1\ldots dt_p \tag{3}$$

where L denotes Lebesgue integral, $y = \theta(x)$ and $k_p \in L_2([a,b]^p)$, what is the relationship between the sets of kernels $\{f_p\}$, and $\{k_p\}$?

In the following we consider problems (P1) and (P3), which are straightforward, and we begin an exploration of the seemingly harder problem (P2). The analysis is based on the structure developed in [4].

We begin with problem (P1). Let $\{\phi_n\}$ be any complete orthonormal set in $\lambda_2(R)$. Then we have for every $t \in T$ (omitting the arguments $t_1,\ldots,t_p$)

$$f_p(t) = \sum a_{n_1\ldots n_k}^{p_1\ldots p_k}(t)\ \phi_{n_1}^{\tilde{\otimes}p_1}\ \tilde{\otimes}\ldots\tilde{\otimes}\ \phi_{n_k}^{\tilde{\otimes}p_k} \tag{4}$$

where the sum is taken over all $k=1,\ldots,p$, $p_1+\ldots+p_k=p$, and $n_1,\ldots,n_k = 1,2,\ldots$ and converges in $\lambda_2(\otimes^p R)$, and the coefficients are given by

$$\begin{aligned} p_1!\ldots p_k!\,a_{n_1\ldots n_k}^{p_1\ldots p_k}(t) &= p!\,\langle f_p(t),\ \phi_{n_1}^{\tilde{\otimes}p_1}\ \tilde{\otimes}\ldots\tilde{\otimes}\ \phi_{n_k}^{\tilde{\otimes}p_k}\rangle_{\lambda_2(\tilde{\otimes}^p R)} \\ &= E\{J_p(f_p(t))J_p(\phi_{n_1}^{\tilde{\otimes}p_1}\ \tilde{\otimes}\ldots\tilde{\otimes}\ \phi_{n_k}^{\tilde{\otimes}p_k})\} \\ &= E\{Y_t H_{p_1}(\textstyle\int\phi_{n_1}(s)X_s ds)\ldots H_{p_n}(\int\phi_{n_k}(s)X_s ds)\}\ . \end{aligned} \tag{5}$$

Hence from the input and output processes X and Y we can find the coefficients a from (5) and thus the kernels $f_p(t)$ from (4). Notice that the functions $\phi_n(t)$ can be chosen by orthonormalizing (in $\lambda_2(R)$ of course) any set of functions complete in $L_2(T)$; or else by putting $\phi_n = \lambda_n^{-\frac{1}{2}} e_n$ where $\{\lambda_n\}$ and $\{e_n\}$ are the eigenvalues and eigenfunctions of $R(t,s)$ [1]. This method of determining f_p has of course all the usual disadvantages of a series expansion. When $T = (-\infty,+\infty)$, X is stationary and $f_p(t;t_1,\ldots,t_p) = f_p(t_1-t, \ldots,t_p-t)$, $p=1,2,\ldots$, then a somewhat simpler method for approximating f_p can be found but we will not go into this here.

Problem (P2) is the most interesting as well as the most difficult one. Of course, if one is willing to put strong assumptions on the system, like those in problem (P3), then, as we shall see, the situation is fairly straightforward. Thus the main point of problem (P2) is to investigate conditions on the system, much weaker than those made in problem (P3), under which appropriate positive answers to problem (P2) can be given. Here we consider only the relationship of the representation of the system for the Gaussian input X to its representation for a deterministic input which is a sample function of X.

From (1) and (4) we have

$$Y_t - E(Y_t) = \sum_{p=1}^{\infty} J_p(f_p)$$

$$= \sum_{p=1}^{\infty} \sum_{\substack{k=1,\ldots,p \\ p_1+\cdots+p_k=p \\ n_1,\ldots,n_k=1,2,\ldots}} a_{n_1\ldots n_k}^{p_1\ldots p_k}(t) J_p\left(\phi_{n_1}^{\tilde{\otimes}p_1} \tilde{\otimes}\ldots\tilde{\otimes} \phi_{n_k}^{\tilde{\otimes}p_k}\right) .$$

Then writing $H_p(\xi) = \sum_{j=0}^{p} c_j^p \xi^p$, for a standard normal random variable ξ, we have

$$J_p(\phi_{n_1}^{\tilde{\otimes}p_1} \tilde{\otimes}\ldots\tilde{\otimes} \phi_{n_k}^{\tilde{\otimes}p_k}) = H_{p_1}(\xi_{n_1})\ldots H_{p_k}(\xi_{n_k})$$

$$= \sum_{\substack{s_i=0,\ldots,p_i \\ i=1,\ldots,k}} c_{j_1}^{p_1}\ldots c_{j_k}^{p_k} \xi_{n_1}^{j_1}\ldots\xi_{n_k}^{j_k}$$

$$= \sum_{\substack{j_i=0,\ldots,p_i \\ i=1,\ldots,k}} c_{j_1}^{p_1}\ldots c_{j_k}^{p_k} L\int\cdots\int (\phi_{n_1}^{\tilde{\otimes}j_1} \tilde{\otimes}\ldots\tilde{\otimes} \phi_{n_k}^{\tilde{\otimes}j_k})(t_1,\ldots,t_j) X_{t_1}\ldots X_{t_j} \, dt_1\ldots dt_j ,$$

where $j=j_1+\ldots+j_k$, and if we define

$$h_q^N(t;t_1,\ldots,t_q) = \sum_{p=q}^{q+N} \sum_{\substack{k=1,\ldots,p \\ n_1,\ldots,n_k=1,\ldots,N}} \sum_{\substack{p_1+\ldots+p_k=p \\ j_1+\ldots+j_k=q}} c_{j_1}^{p_1}\ldots c_{j_k}^{p_k} a_{n_1\ldots n_k}^{p_1\ldots p_k}(t) \left(\phi_{n_1}^{\tilde{\otimes}j_1} \tilde{\otimes}\ldots\tilde{\otimes} \phi_{n_k}^{\tilde{\otimes}j_k}\right)(t_1,\ldots,t_q)$$

we obtain by a simple rearrangement of terms

$$Y_t - E(Y_t) = \sum_{q=0}^{\infty} \lim_{N\to\infty} L\int\cdots\int h_q^N(t;t_1,\ldots,t_q)X_{t_1}\ldots X_{t_q}dt_1\ldots dt_q \tag{6}$$

Since the convergence in (6) is in mean square, along some subsequence we will have convergence with probability one. Thus we can write

$$Y_t - E(Y_t) = \lim_{n\to\infty} \sum_{q=0}^{N_n} L\int\cdots\int h_q^{N_n}(t;t_1,\ldots,t_q)X_{t_1}\ldots X_{t_q}dt_1\ldots dt_q \quad \text{a.s.} \tag{7}$$

Thus for almost every sample function of the process X, the output of the system can be represented by (7). Notice that the kernels h_q^N in (7) can be found from the kernels f_p and that $h_q^N(t;t_1,\ldots,t_p)$ are continuous functions in $t_1,\ldots,t_p$ if we choose (as we can) the functions $\phi_n(t)$ to be continuous. Hence knowing the representation of the system output to a Gaussian input we can find the representation of the system output for a certain class of deterministic inputs, namely almost all sample functions of the Gaussian process. This deterministic input-output representation, given by (7), depends of course on the covariance R of the Gaussian process in the following two ways:

(i) R determines the kernels h_q^N of the representation (7), and

(ii) R determines, up to a zero probability set, the deterministic functions for which representation (7) is valid.

Representation (7) has the following two remarkable features. First, even though it is valid for a small class of deterministic inputs and its kernels depend on that class, it was obtained under extremely weak assumptions on the system, namely $E(Y_t^2) < \infty$. Second it is identical in form to the representation

$$y(t) = \lim_{n\to\infty}\{k_0^{N_n}(t) + \sum_{q=1}^{N_n} L\int\cdots\int k_q^{N_n}(t;t_1,\ldots,t_q)x(t_1)\ldots x(t_q)dt_1\ldots dt_q\} \tag{8}$$

obtained by Fréchet (1910) for all x(t) in C[a,b] or in $L_2[a,b]$ under the assumption that the system is continuous, in the sense that for each fixed t, y(t) is a continuous functional on C[a,b] or $L_2[a,b]$; in (8) the kernels k_q^N depend only on the system and not on the particular input x(t) in C[a,b] or in $L_2[a,b]$. It is thus remarkable that a representation like (8), valid not for all but only for some functions in $L_2[a,b]$ (or in C[a,b] if X has continuous sample functions), was derived with no continuity assumptions on the system.

If, furthermore, it turns out that for each q=1,2,..., the kernels $h_q^N(t)$ converge in $L_2([a,b]^q)$ to $h_q(t)$ (which constitutes an additional restriction on the system) then (6) and thus (7) may be simplified to

$$Y_t - E(Y_t) = \lim_{n\to\infty} \sum_{q=0}^{N_n} L\int\cdots\int h_q(t;t_1,\ldots,t_q)X_{t_1}\ldots X_{t_q}dt_1\ldots dt_q \quad \text{a.s.} \tag{9}$$

Finally, we should remark that if the system acting on the Gaussian input X is of finite order P, in the sense that in (1) $f_p = 0$ for $p > P$,

$$Y_t - E(Y_t) = \sum_{p=1}^{P} \int\cdots\int f_p(t;t_1,\ldots,t_p)X_{t_1}\ldots X_{t_p}dt_1\ldots dt_p\ , \tag{10}$$

then it has the same order P when acting on the sample functions of X, in the sense that (7) is written as

$$Y_t - E(Y_t) = \lim_{n\to\infty} \sum_{p=0}^{P} \int\cdots\int h_p^{N^n}(t;t_1,\ldots,t_p)X_{t_1}\ldots X_{t_p}dt_1\ldots dt_p \quad \text{a.s.} \tag{11}$$

and if, furthermore, each $h_p^N(t)$ converges in $L_2([a,b]^p)$ to $h_p(t)$ we have

$$Y_t - E(Y_t) = \sum_{p=0}^{P} \int\cdots\int h_p(t;t_1,\ldots,t_p)X_{t_1}\ldots X_{t_p}dt_1\ldots dt_p \quad \text{a.s.} \tag{12}$$

It may be checked (even though the calculations are somewhat messy) that in the latter case we always have

$$h_P = f_P \quad \text{and} \quad h_{P-1} = f_{P-1}$$

the assumption on the convergence of the $h_p^N(t)$'s implying that the kernels $f_p(t)$ belong to $L_2([a,b]^p)$.

We finally consider problem (P3) which consists in finding the kernels $\{k_p\}$ when the kernels $\{f_p\}$ are known, and vice versa. From (3) we have

$$Y_t = k_0(t) + \sum_{p=1}^{\infty} L\int\cdots\int k_p(t;t_1,\ldots,t_p)X_{t_1}\ldots X_{t_p}dt_1\ldots dt_p \quad \text{a.s.} \tag{13}$$

On the other hand representation (1) implies (7) which in view of (13) can be written as in (9). Now it follows from (9) and (13) that

$$k_0(t) = E(Y_t) + h_0(t)$$

and for p=1,2,...

$$\begin{aligned} &L\int\cdots\int k_p(t;t_1,\ldots,t_p)X_{t_1}\ldots X_{t_p}dt_1\ldots dt_p \\ &\quad = L\int\cdots\int h_p(t;t_1,\ldots,t_p)X_{t_1}\ldots X_{t_p}dt_1\ldots dt_p \quad \text{a.s.} \end{aligned} \tag{14}$$

If M_X^p is the subspace of $L_2([a,b]^p)$ generated by the symmetric functions $\{X_{t_1}(\omega)\ldots X_{t_p}(\omega),\ \omega\in\Omega-\Omega_0\}$ where Ω_0 is the exceptional set in (14), then (14) is equivalent to

$$h_p(t) = \text{Proj.}_{M_X^p} k_p(t)\ .$$

Thus, in general, knowledge of the kernels $\{f_p\}$, and thus the kernels $\{h_p\}$, determines only the projections of the kernels k_p onto M_X^p. The kernels k_p will be determined from the kernels f_p, by $k_p = h_p$, only if the subspaces

M_X^p consist of all symmetric functions in $L_2([a,b]^p)$. An equivalent condition is that if

$$R_p(t_1,\ldots,t_p;s_1,\ldots,s_p) = E(X_{t_1}\ldots X_{t_p} X_{s_1}\ldots X_{s_p})$$

and R_p denotes also the integral type operator in $L_2([a,b]^p)$ with kernel R_p, then R_p should be strictly positive definite, or the null space of $R_p^{\frac{1}{2}}$ should be $\{0\}$.

Conversely, knowledge of the kernels k_p clearly determines the kernels f_p. This is quite obvious from (13). The precise relationship can be derived by using the property

$$\begin{aligned}\int\cdots\int f_p(t;t_1,\ldots,t_p)X_{t_1}\ldots X_{t_p}dt_1\ldots dt_p &= \mathrm{Proj}_{\overline{\mathcal{Q}}_p} Y_t \\ &= \sum_{q=p}^{\infty} L\int\cdots\int k_p(t;t_1,\ldots,t_q)\mathrm{Proj}_{\overline{\mathcal{Q}}_p}(X_{t_1}\ldots X_{t_q})dt_1\ldots dt_q\end{aligned} \qquad (15)$$

where $\overline{\mathcal{Q}}_p$ is the closure in $L_2(X)$ of $\mathcal{Q}_p$, the set of all polynomials in the elements of $H(X)$ with degree p which are orthogonal to all polynomials with degree $\le p-1$. The projections of $(X_{t_1}\ldots X_{t_q})$ onto $\overline{\mathcal{Q}}_p$ can be expressed by using the Grad-Barrett-Hermite polynomials (see Section II of Root (1965)) and then (15) will give each f_p in terms of k_p, k_{p+2}, ... When only the first P terms in (13) are present, the same will be true for (1) and in this case f_P can be expressed in terms of k_P, f_{P-1} in terms of k_{P-1}, f_{P-2} in terms of k_{P-2} and k_P, f_{P-3} in terms of k_{P-3} and k_{P-1}, etc. Again we will have in fact $f_P = k_P$ and $f_{P-1} = k_{P-1}$ as shown in Root (1965), where the specific expressions for $P = 3$ are also given.

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